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INTEGRAL OPERATORS AND INTEGRAL COHOMOLOGY CLASSES OF HILBERT SCHEMES

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Abstract. The methods of integral operators on the cohomology of Hilbert schemes of points on surfaces are developed. They are used to establish integral bases for the cohomology groups of Hilbert schemes of points on a class of surfaces (and conjecturally, for all simply connected surfaces).

1. Introduction

The Hilbert scheme $X^{[n]}$ parameterizes all length-n 0-dimensional closed subscheme of a complex smooth projective surface X. A classical result of Göttsche [Got] calculated the Betti numbers of the Hilbert scheme $X^{[n]}$ for an arbitrary surface X. Nakajima [Na1] (see also [Gro]) constructed a Heisenberg algebra which acts irreducibly on the direct sum $\mathbb{H}_X$ of the rational cohomology of the Hilbert schemes $X^{[n]}$ for all n. As a corollary, a linear basis for the *rational* cohomology of $X^{[n]}$ in terms of the Heisenberg algebra operators can be constructed. With the help of this fundamental construction, there has been intensive activities and significant progress in the study of Hilbert schemes such as their rational cohomology rings made by Lehn, Sorger, W.-P. Li and the authors of this paper, and others (cf. [QW] for extensive references). However, there has been virtually no work toward the basic problem of studying the *integral* cohomology of $X^{[n]}$ for a general surface X (on the other hand, see [ES2, LS] when X is the projective or affine plane).

The purpose of this paper is to develop some general techniques of integral operators and then to use them to find integral bases for the *integral* cohomology of $X^{[n]}$ for a certain class of surfaces X. By an integral operator we mean a linear operator on $\mathbb{H}_X$ which sends every integral cohomology class to an integral one. One of the difficulties of studying the integral cohomology of $X^{[n]}$ is that not many rational cohomology classes are known to be integral. The starting point of this work is the following key observation which we derive from the Stability Theorem established in [LQW]: if A is an integral cohomology class of $X^{[n]}$ and if A is written as $\mathfrak{a}_A|0\rangle$ where $\mathfrak{a}_A$ is a polynomial of creation Heisenberg operators, then $\mathfrak{a}_A$ is an integral operator (Proposition 3.5). This provides an effective method of constructing new integral classes from known ones.

Our study of integral operators and integral classes are roughly divided into two distinct parts. The first part involves integral operators and integral classes that

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are 'created' from the creation Heisenberg operators $\mathfrak{a}_{-r}(1)$ and $\mathfrak{a}_{-r}(x)$ associated to the identity cohomology class 1 and the point cohomology class x. For a class α in the rational cohomology $H^*(X)$ and for a partition $\lambda = (1^{m_1}2^{m_2}\ldots)$ where m_r stands for the number of parts equal to r, define $|\lambda| = \sum_{r\ge1} rm_r$, $\ell(\lambda) = \sum_{r\ge1} m_r$,

$$\mathfrak{z}_\lambda = \prod_{r\ge1} r^{m_r} m_r!, \quad \mathfrak{a}_{-\lambda}(\alpha) = \prod_{r\ge1} \mathfrak{a}_{-r}(\alpha)^{m_r}. \tag{1.1}$$

We prove that the operator $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)$ is integral. By construction, the operator $\mathfrak{a}_{-\lambda}(x)$ associated to the point class x is always integral.

The second part of our study is to develop integral operators and integral classes created from the creation Heisenberg operators $\mathfrak{a}_{-i}(\alpha)$ associated with $\alpha \in H^2(X)$. Recall from [Na2] (see also [Gro]) the subvarieties $L^\lambda C \subset X^{[n]}$ associated to partitions λ of n and an embedded curve C (see Subsection 4.1 for definitions). Let $\mathfrak{m}_{\lambda,C}$ be the integral operator $\mathfrak{a}_{[L^\lambda C]}$. We first extend the definitions of $[L^\lambda C]$ and $\mathfrak{m}_{\lambda,C}$ to $[L^\lambda \alpha]$ and $\mathfrak{m}_{\lambda,\alpha}$ for an arbitrary class $\alpha \in H^2(X)$. We prove that $\mathfrak{m}_{\lambda,\alpha}$ is an integral operator for $\alpha = \alpha_1 \pm \alpha_2$ whenever $\mathfrak{m}_{\lambda,\alpha_1}$ and $\mathfrak{m}_{\lambda,\alpha_2}$ are integral. In particular, it follows that $\mathfrak{m}_{\lambda,\alpha}$ is integral for every divisor α (Theorem 4.5). Moreover, we show (cf. Theorem 4.6) that if the intersection matrix of $\alpha_1, \cdots, \alpha_k$ in $H^2(X)$ has determinant ±1, then so is the determinant of the intersection matrix of the classes $\mathfrak{m}_{\nu^1,\alpha_1} \cdots \mathfrak{m}_{\nu^k,\alpha_k}|0\rangle$ in $H^*(X^{[n]})$, where the partitions ν^i satisfy $|\nu^1| + \cdots + |\nu^k| = n$. Our proofs of these results use in an effective way the interrelations among the operators $\mathfrak{m}_{\nu,\alpha}$, the creation Heisenberg operators, and the monomial symmetric functions m_ν in the ring of symmetric functions (cf. Chapter 9, [Na2]).

As an application of the above development, we obtain the following.

Theorem 1.1. *Let X be a projective surface with $H^1(X;\mathcal{O}_X) = H^2(X;\mathcal{O}_X) = 0$. Let $\alpha_1, \cdots, \alpha_k$ be an integral basis for $H^2(X;\mathbb{Z})/\mathrm{Tor}$. Then the following classes*

$$\frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)\mathfrak{m}_{\nu^1,\alpha_1} \cdots \mathfrak{m}_{\nu^k,\alpha_k}|0\rangle, \quad |\lambda| + |\mu| + \sum_{i=1}^k |\nu^i| = n$$

are integral, and furthermore, they form an integral basis for $H^*(X^{[n]};\mathbb{Z})/\mathrm{Tor}$.

We conjecture that the cohomology class $[L^\lambda \alpha]$ is integral whenever $\alpha \in H^2(X)$ is an integral class. If this conjecture is true, then the statement in Theorem 1.1 will be valid for every simply connected projective surface X. We refer to Remark 5.6 for some discussions in this direction.

In view of the dictionary between Hilbert schemes and symmetric products developed in [QW], the counterparts of the results in this paper are expected to be valid for the Chen-Ruan orbifold cohomology rings of the symmetric products.

Our paper is organized as follows. In Sect. 2, we review some basic results about the Hilbert schemes. In Sect. 3 and Sect. 4, integral operators related to the creation operators $\mathfrak{a}_{-i}(1)$ and $\mathfrak{a}_{-i}(\alpha)$ with $\alpha \in H^2(X)$ are investigated respectively. In Sect. 5, we prove Theorem 1.1 and some other general structure results.

Conventions. Unless otherwise indicated, all the cohomology groups in this paper are in $\mathbb{Q}$-coefficients. For a continuous map $p : Y_1 \to Y_2$ between two smooth

compact manifolds and for $\alpha_1 \in H^*(Y_1)$, we define $p_*(\alpha_1)$ to be $\mathrm{PD}^{-1}p_*(\mathrm{PD}(\alpha_1))$ where PD stands for the Poincaré duality. For a smooth projective surface X, by abusing notations, we let $1 \in H^0(X;\mathbb{Z})$ be the fundamental cohomology class of X; in addition, we let x denote either a point in X, or the class in $H^4(X;\mathbb{Z})$ which is the Poincaré dual of the homology class represented by a point in X.

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2. Basics on Hilbert schemes of points on surfaces

Let X be a complex smooth projective surface, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ be the corresponding sheaf of ideals. It is well known that $X^{[n]}$ is smooth. Sending an element in $X^{[n]}$ to its support in the symmetric product $\mathrm{Sym}^n(X)$, we obtain the Hilbert-Chow morphism $\pi_n : X^{[n]} \to \mathrm{Sym}^n(X)$, which is a resolution of singularities. Let $\mathscr{Z}_n$ be the universal codimension-2 subscheme of $X^{[n]} \times X$, which can be described set-theoretically by

$$\mathscr{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \,|\, x \in \mathrm{Supp}(\xi)\} \subset X^{[n]} \times X. \tag{2.1}$$

Let $H^*(X^{[n]})$ be the total cohomology of $X^{[n]}$ with $\mathbb{Q}$-coefficients, and put

$$\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}). \tag{2.2}$$

For $m \ge 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$, and let $Q^{[m+n,m]}$ be the closed subscheme of $X^{[m+n]} \times X \times X^{[m]}$ defined in [Lehn] whose set-theoretical description is:

$$\{(\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \,|\, \xi \supset \eta \text{ and } \mathrm{Supp}(I_\eta/I_\xi) = \{x\}\}.$$

We recall Nakajima's definition of the Heisenberg operators [Na1]. Let $n > 0$. The linear operator $\mathfrak{a}_{-n}(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ is defined by

$$\mathfrak{a}_{-n}(\alpha)(A) = \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^*\alpha \cdot \tilde{p}_2^*A) \tag{2.3}$$

for $A \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Define $\mathfrak{a}_n(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ to be $(-1)^n$ times the operator obtained from the definition of $\mathfrak{a}_{-n}(\alpha)$ by switching the roles of $\tilde{p}_1$ and $\tilde{p}_2$. We often refer to $\mathfrak{a}_{-n}(\alpha)$ (resp. $\mathfrak{a}_n(\alpha)$) as the *creation* (resp. *annihilation*) operator. We also set $\mathfrak{a}_0(\alpha) = 0$. A non-degenerate super-symmetric bilinear form $(-,-)$ on $\mathbb{H}_X$ is induced from the standard one on $H^*(X^{[n]})$ defined by

$$(\alpha, \beta) = \int_{X^{[n]}} \alpha\beta, \qquad \alpha, \beta \in H^*(X^{[n]}).$$

This allows us to define the *adjoint* $\mathfrak{f}^\dagger \in \mathrm{End}(\mathbb{H}_X)$ for $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$. Then,

$$\mathfrak{a}_n(\alpha) = (-1)^n \cdot \mathfrak{a}_{-n}(\alpha)^\dagger. \tag{2.4}$$

The operators $\mathfrak{a}_n(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ and $n \in \mathbb{Z}$ satisfy the following Heisenberg algebra commutation relation (cf. [Na2]):

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m,-n} \cdot (\alpha, \beta) \cdot \mathrm{Id}_{\mathbb{H}_X}. \tag{2.5}$$

The space $\mathbb{H}_X$ is an irreducible module over the Heisenberg algebra generated by the operators $\mathfrak{a}_n(\alpha)$ with a highest weight vector

$$|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{Q}.$$

It follows that $\mathbb{H}_X$ is linearly spanned by all the *Heisenberg monomial classes*:

$$\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle \tag{2.6}$$

where $k \geq 0, n_1, \ldots, n_k > 0$, and $\alpha_1, \ldots, \alpha_k$ run over a linear basis of $H^*(X)$.

For a nonnegative integer n, we define the operator:

$$\mathbf{1}_{-n} = \frac{1}{n!} \cdot \mathfrak{a}_{-(1^n)}(1) = \frac{1}{n!} \cdot \mathfrak{a}_{-1}(1)^n. \tag{2.7}$$

The geometric meaning of $\mathbf{1}_{-n}$ is that $\mathbf{1}_{-n}|0\rangle$ is equal to the fundamental class of the Hilbert scheme $X^{[n]}$. For simplicity, we set $\mathbf{1}_{-n} = 0$ when $n < 0$.

The following is the Stability Theorem 5.1 proved in [LQW].

Theorem 2.1. *Let $s \geq 1$ and $k_i \geq 1$ for $1 \leq i \leq s$. Fix $n_{i,j} \geq 1$ and $\alpha_{i,j} \in H^*(X)$ for $1 \leq j \leq k_i$, and fix n with $n \geq \sum_{j=1}^{k_i} n_{i,j}$ for all $1 \leq i \leq s$. Then the cup product*

$$\prod_{i=1}^{s} \left(\mathbf{1}_{-(n-\sum_{j=1}^{k_i} n_{i,j})} \left(\prod_{j=1}^{k_i} \mathfrak{a}_{-n_{i,j}}(\alpha_{i,j}) \right) |0\rangle \right) \tag{2.8}$$

in $H^*(X^{[n]})$ is equal to a finite linear combination of monomials of the form

$$\mathbf{1}_{-(n-\sum_{p=1}^{N} m_p)} \left(\prod_{p=1}^{N} \mathfrak{a}_{-m_p}(\gamma_p) \right) |0\rangle \tag{2.9}$$

whose coefficients are independent of $X, \alpha_{i,j}$ and n. Here $\sum_{p=1}^{N} m_p \leq \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j}$, and $\gamma_1, \ldots, \gamma_N$ depend only on $\alpha_{i,j}$, $1 \leq i \leq s, 1 \leq j \leq k_i$, and the canonical class and the Euler class of X. In addition, the expression (2.9) satisfies the upper bound

$$\sum_{p=1}^{N} m_p = \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j}$$

if and only if it is $\mathbf{1}_{-(n-\sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j})} \left(\prod_{i=1}^{s} \prod_{j=1}^{k_i} \mathfrak{a}_{-n_{i,j}}(\alpha_{i,j}) \right) |0\rangle$ with coefficient 1.

3. Integral operators involving only $1 \in H^*(X)$

3.1. Integral operators.

Definition 3.1. (i) A class $A \in H^*(X^{[n]})$ is *integral* if it is contained in

$$H^*(X^{[n]}; \mathbb{Z})/\mathrm{Tor} \subset H^*(X^{[n]});$$

(ii) A linear basis of $H^*(X^{[n]})$ is *integral* if its members are integral classes and form a $\mathbb{Z}$-basis of the lattice $H^*(X^{[n]}; \mathbb{Z})/\mathrm{Tor}$;

(iii) A linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$ is *integral* if $\mathfrak{f}(A) \in \mathbb{H}_X$ is an integral class whenever $A \in \mathbb{H}_X$ is an integral cohomology class.

A linear basis of $H^*(X^{[n]})$ is integral if and only if its members are integral classes and the matrix formed by the pairings of its members is unimodular.

Lemma 3.2. (i) *If $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$ is integral, then so is its adjoint $\mathfrak{f}^\dagger$;*

(ii) *The Heisenberg operators $\mathfrak{a}_n(\alpha), n \in \mathbb{Z}$ are integral if $\alpha \in H^*(X)$ is integral.*

Proof. (i) Note that a class $A \in \mathbb{H}_X$ is integral if and only if (A, B) is an integer whenever $B \in \mathbb{H}_X$ is an integral class. It follows that $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$ is integral if and only if its adjoint operator $\mathfrak{f}^\dagger \in \mathrm{End}(\mathbb{H}_X)$ is integral.

(ii) Recall that $\mathfrak{a}_0(\alpha) = 0$. Next, fix $n > 0$. By (2.3), the Heisenberg operator $\mathfrak{a}_{-n}(\alpha)$ is integral. By (2.4) and (i), the operator $\mathfrak{a}_n(\alpha)$ is integral as well. $\square$

Lemma 3.3. *For $n \ge 0$, the operator $\mathbf{1}_{-n}$ is integral.*

Proof. Recall from (2.7) that $\mathbf{1}_{-n} = 1/n! \cdot \mathfrak{a}_{-1}(1)^n$. Fix any integer $m \ge 0$ and an integral class $A \in H^j(X^{[m]})$. For $i = m, \ldots, m+n-1$, let Q_i be the image of the subscheme $Q^{[i+1,i]}$ under the natural projection $X^{[i+1]} \times X \times X^{[i]} \to X^{[i+1]} \times X^{[i]}$. Set-theoretically, $Q_i = \{(\xi_{m+1}, \xi_m) \in X^{[i+1]} \times X^{[i]} \,|\, \xi_{m+1} \supset \xi_m\}$. Let $\phi_{i,1}$ and $\phi_{i,2}$ be the two projections of $X^{[i+1]} \times X^{[i]}$. It follows from (2.3) that

$$\mathfrak{a}_{-1}(1)(A) = (\phi_{m,1})_*([Q_m] \cdot \phi_{m,2}^* A).$$

Repeating this process and using the projection formula, we conclude that

$$\mathfrak{a}_{-1}(1)^n(A) = (\tilde{\phi}_{m+n})_*([Q] \cdot \tilde{\phi}_m^* A)$$

where $\tilde{\phi}_i$ denotes the projection of $Y := X^{[m+n]} \times \cdots \times X^{[m+1]} \times X^{[m]}$ to $X^{[i]}$ and

$$Q = \{(\xi_{m+n}, \ldots, \xi_{m+1}, \xi_m) \in Y |\ \xi_{m+n} \supset \cdots \supset \xi_{m+1} \supset \xi_m\}$$

(the scheme structure on Q can be described similarly as that on Q_i). To show that $\mathbf{1}_{-n}(A) \in H^j(X^{[m+n]})$ is integral, it suffices to prove that the intersection number $(\tilde{\phi}_{m+n})_*([Q] \cdot \tilde{\phi}_m^* A) \cdot B$ is divisible by $n!$ for any integral class $B \in H^{4(m+n)-j}(X^{[m+n]})$.

Represent the integral class A by a piecewise smooth cycle $W_A \subset X^{[m]}$. Then, the integral class $[Q] \cdot \tilde{\phi}_m^* A$ is represented by

$$Q_A := \{(\xi_{m+n}, \ldots, \xi_{m+1}, \xi_m) \in Y |\ \xi_{m+n} \supset \cdots \supset \xi_{m+1} \supset \xi_m \text{ and } \xi_m \in W_A\}.$$

Note that $\tilde{\phi}_{m+n}|_{Q_A} : Q_A \to \tilde{\phi}_{m+n}(Q_A)$ is generically finite, and a generic element in $\tilde{\phi}_{m+n}(Q_A)$ is of the form $\xi_m + x_1 + \ldots + x_n$ where $\xi_m \in W_A$ is generic, the points $x_1, \ldots, x_n$ are distinct, and $\mathrm{Supp}(\xi_m) \cap \{x_1, \ldots, x_n\} = \emptyset$.

Represent B by a piecewise smooth cycle W_B such that W_B intersects $\tilde{\phi}_{m+n}(Q_A)$ transversely at generic points $P_1, \ldots, P_s$. Write P_i as $\xi_m + x_1 + \ldots + x_n$. Let

$$\xi_m = \eta_1 + \ldots + \eta_t + x_1' + \ldots + x_u' + x_1'' + \ldots + x_v''$$

where each η_i is supported at one point with $\ell(\eta_i) \geq 2$, the following subsets of X:

$$\mathrm{Supp}(\eta_1), \ldots, \mathrm{Supp}(\eta_t), \{x_1'\}, \ldots, \{x_u'\}, \{x_1''\}, \ldots, \{x_v''\}$$

are mutually disjoint, and only the points $x_1', \ldots, x_u'$ can move in Zariski open subsets of the surface X. (In other words, if we put

$$U = X - \mathrm{Supp}(\eta_1) \cup \ldots \cup \mathrm{Supp}(\eta_t) \cup \{x_1', \ldots, x_u', x_1'', \ldots, x_v''\},$$

then for all distinct points $\tilde{x}_1', \ldots, \tilde{x}_u'$ in U, we have

$$\eta_1 + \ldots + \eta_t + \tilde{x}_1' + \ldots + \tilde{x}_u' + x_1'' + \ldots + x_v'' \in W_A.)$$

So $\left(\tilde{\phi}_{m+n}|_{Q_A}\right)^{-1}(P_i)$ consists of $(u+n)(u+n-1)\cdots(u+1)$ distinct points. It follows that the intersection number $(\tilde{\phi}_{m+n})_*([Q] \cdot \tilde{\phi}_m^* A) \cdot B$ is divisible by $n!$. $\square$

Let $A \in H^*(X^{[n]})$. By (2.6), A is a linear combination of classes of the form $\mathfrak{a}_{-1}(1)^m \mathfrak{a}_{-m_1}(\alpha_1) \cdots \mathfrak{a}_{-m_\ell}(\alpha_\ell)|0\rangle$ where $\alpha_1, \ldots, \alpha_\ell \in \bigoplus_{i \geq 1} H^i(X)$, $m \geq 0$, $\ell \geq 0$, and $m_1, \ldots, m_\ell > 0$. By (2.5), $\mathfrak{a}_1(x)\big(\mathfrak{a}_{-m_1}(\alpha_1) \cdots \mathfrak{a}_{-m_\ell}(\alpha_\ell)|0\rangle\big) = 0$. It follows that the class $A \in H^*(X^{[n]})$ can be written as

$$A = \mathbf{1}_{-n_1}(A_1) + \ldots + \mathbf{1}_{-n_k}(A_k) \tag{3.1}$$

where $0 \leq n_1 < \ldots < n_k$, and $A_i \in H^*(X^{[n-n_i]})$ with $\mathfrak{a}_1(x)(A_i) = 0$ for every i.

Lemma 3.4. *Let $A \in H^*(X^{[n]})$ be expressed as in (3.1). Then,*

(i) *the classes A_i are uniquely determined by A;*
(ii) *A is integral if and only if all the classes A_i are integral.*

Proof. First of all, by Lemma 3.3, A is integral if all the classes A_i are integral. So it remains to prove (i) and the "only if" part of (ii).

Next, applying the operator $\mathfrak{a}_1(x)^{n_k}$ to both sides of (3.1), we obtain

$$\mathfrak{a}_1(x)^{n_k}(A) = \mathfrak{a}_1(x)^{n_k}\big(\mathbf{1}_{-n_1}(A_1) + \ldots + \mathbf{1}_{-n_k}(A_k)\big) = (-1)^{n_k} \cdot A_k.$$

Thus the class A_k is uniquely determined by A. Moreover, by Lemma 3.2 (ii), if A is an integral class, then so is the class A_k.

Finally, repeating the above process to the class $A' := (A - \mathbf{1}_{-n_k}(A_k))$, we conclude that (i) and the "only if" part of (ii) hold for all the classes A_i. $\square$

Proposition 3.5. *Let $A \in \mathbb{H}_X$ be an integral class. Write A as $\mathfrak{a}_A|0\rangle$ where $\mathfrak{a}_A$ is a unique polynomial of creation operators. Then, $\mathfrak{a}_A$ is an integral operator.*

Proof. Let $A \in H^*(X^{[n]})$. Fix an integral class $B \in H^*(X^{[m]})$. We want to show that the cohomology class $\mathfrak{a}_A(B) \in H^*(X^{[m+n]})$ is still integral.

First of all, decomposing A as in (3.1), we see from Lemma 3.4 (ii) and Lemma 3.3 that we may assume $\mathfrak{a}_1(x)(A) = 0$. Similarly, write $B = \mathbf{1}_{-m_1}(B_1) + \ldots + \mathbf{1}_{-m_k}(B_k)$

as in (3.1), where $0 \le m_1 < \ldots < m_k \le m$, and $B_i \in H^*(X^{[m-m_i]})$ with $\mathfrak{a}_1(x)(B_i) = 0$ for every i. By Lemma 3.4 (ii), each class B_i is integral. Now

$$\mathfrak{a}_A(B) = \sum_{i=1}^k \mathfrak{a}_A\big(\mathbf{1}_{-m_i}(B_i)\big) = \sum_{i=1}^k \mathbf{1}_{-m_i}\big(\mathfrak{a}_A(B_i)\big).$$

It follows from Lemma 3.3 that we may assume $\mathfrak{a}_1(x)(B) = 0$ as well.

By Lemma 3.3 again, we have two integral classes $\mathbf{1}_{-m}A, \mathbf{1}_{-n}B \in H^*(X^{[m+n]})$. By Theorem 2.1, the cup product $(\mathbf{1}_{-m}A) \cdot (\mathbf{1}_{-n}B)$ equals

$$\mathfrak{a}_A(B) + \sum_{i \ge 1} \mathbf{1}_{-i}(A_i)$$

where $\mathfrak{a}_1(x)(A_i) = 0$ for every $i \ge 1$. By our assumption on A and B, we have $\mathfrak{a}_1(x)\big(\mathfrak{a}_A(B)\big) = 0$. Hence $\mathfrak{a}_A(B) \in H^*(X^{[m+n]})$ is integral by Lemma 3.4. $\square$

3.2. **Integral classes and operators involving only $1 \in H^*(X)$.**

In view of Proposition 3.5, to obtain integral operators, we need to construct integral classes. We begin with the Chern classes of some tautological vector bundles over $X^{[n]}$. For a line bundle L on the surface X, let $L^{[n]} = \big(p_1|_{\mathcal{Z}_n}\big)_*\big(p_2|_{\mathcal{Z}_n}\big)^*L$ where $\mathcal{Z}_n$ is from (2.1), and p_1 and p_2 are the projections of $X^{[n]} \times X$ to its two factors. By the Theorem 4.6 in [Lehn] which was conjectured earlier by Göttsche,

$$c_i(\mathcal{O}_X^{[n]}) = (-1)^i \cdot \sum_{|\nu|=n,\ \ell(\nu)=n-i} \frac{\mathfrak{a}_{-\nu}(1)|0\rangle}{\mathfrak{z}_\nu}. \tag{3.2}$$

Lemma 3.6. *For every partition λ, we have*

(i) $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)|0\rangle$ *is an integral class;*

(ii) $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)$ *is an integral operator.*

Proof. (ii) follows from (i) and Proposition 3.5. To prove (i), we let $n = |\lambda|$ and use induction on n. Our result is trivially true when $n = 0, 1$. In the following, assume that $1/\mathfrak{z}_\mu \cdot \mathfrak{a}_{-\mu}(1)|0\rangle$ is integral whenever $|\mu| < n$.

Let $\lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots)$ so that $n = \sum_r rm_r$. First of all, assume $m_r > 0$ for at least two different r's. Then, $rm_r < n$ for every r. Putting $\mu^r = (r^{m_r})$ for $r \ge 1$ and applying induction to the partitions μ^r, we obtain integral classes $A_r := 1/\mathfrak{z}_{\mu^r} \cdot \mathfrak{a}_{-\mu^r}(1)|0\rangle$. By Proposition 3.5, the operators $\mathfrak{a}_{A_r} = 1/\mathfrak{z}_{\mu^r} \cdot \mathfrak{a}_{-\mu^r}(1)$ are integral. Thus, $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)|0\rangle = \prod_{r\ge1}\big(1/\mathfrak{z}_{\mu^r} \cdot \mathfrak{a}_{-\mu^r}(1)\big) \cdot |0\rangle$ is integral.

We are left with the case when $m_r > 0$ for a unique r. In this case, $|\lambda| = rm_r = n$ and $\ell(\lambda) = m_r$. Applying (3.2) to $i = n - m_r$, we have

$$\begin{aligned} c_i(\mathcal{O}_X^{[n]}) &= (-1)^i \cdot \sum_{|\nu|=rm_r,\ \ell(\nu)=m_r} \frac{\mathfrak{a}_{-\nu}(1)|0\rangle}{\mathfrak{z}_\nu} \\ &= (-1)^i \cdot \frac{\mathfrak{a}_{-\lambda}(1)|0\rangle}{\mathfrak{z}_\lambda} + (-1)^i \cdot \sum_{\substack{\nu\ne\lambda, |\nu|=rm_r \\ \ell(\nu)=m_r}} \frac{\mathfrak{a}_{-\nu}(1)|0\rangle}{\mathfrak{z}_\nu}. \end{aligned} \tag{3.3}$$

Note that for any partition $\nu = (1^{t_1}2^{t_2}\cdots)$ satisfying $\nu \neq \lambda, |\nu| = rm_r$ and $\ell(\nu) = m_r$, there are at least two i's with $t_i > 0$. By the previous paragraph, $1/\mathfrak{z}_\nu \cdot \mathfrak{a}_{-\nu}(1)|0\rangle$ is integral. Hence we see from (3.3) that $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)|0\rangle$ is integral. $\square$

4. Integral operators involving only classes in $H^2(X)$

4.1. Integral classes involving only an embedded curve in X.

Let C be a smooth irreducible curve in the surface X. By abusing notations, we also use C to denote the corresponding divisor and the corresponding cohomology class. Following Subsection 9.3 of [Na2] (see also [Gro]), for every partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, we define the subvariety $L^\lambda C := \overline{(\pi_n)^{-1}(S^n_\lambda C)}$ of $X^{[n]}$, where $S^n_\lambda C = \{\sum_i \lambda_i x_i | \ x_i \in C, x_i \neq x_j \text{ for } i \neq j\}$, and π_n is the Hilbert-Chow morphism. For $n \geq 0$, let $\mathbb{H}_{n,C}$ be the $\mathbb{Q}$-linear span of all the classes $\mathfrak{a}_{-\lambda}(C)|0\rangle, \lambda \vdash n$ where $\lambda \vdash n$ denotes that λ is a partition of n. By the Theorem 9.14 in [Na2], the integral class $[L^\lambda C] \in H^{2n}(X^{[n]}; \mathbb{Z})$ is contained in $\mathbb{H}_{n,C} \subset H^*(X^{[n]})$. Define

$$\mathbb{H}_C = \bigoplus_{n=0}^{\infty} \mathbb{H}_{n,C}. \tag{4.1}$$

Let Λ be the ring of symmetric functions in infinitely many variables (see p.19 of [Mac]), and $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$. Let Λ^n and $\Lambda^n_\mathbb{Q}$ be the degree-n parts in Λ and $\Lambda_\mathbb{Q}$ respectively. For a partition λ, let p_λ, m_λ and s_λ be the power-sum symmetric function, the monomial symmetric function and the Schur function respectively.

In [Na2], Nakajima defined a linear isomorphism

$$\Phi_C : \quad \Lambda_\mathbb{Q} = \bigoplus_{n=0}^{\infty} \Lambda^n_\mathbb{Q} \quad \to \quad \mathbb{H}_C \tag{4.2}$$

which satisfies the following two properties:

$$\Phi_C(p_\lambda) = \mathfrak{a}_{-\lambda}(C)|0\rangle, \qquad \Phi_C(m_\lambda) = [L^\lambda C]. \tag{4.3}$$

For a partition λ, define $\mathfrak{m}_{\lambda,C} = \mathfrak{a}_{[L^\lambda C]} \in \mathrm{End}(\mathbb{H}_X)$. By Proposition 3.5, the operator $\mathfrak{m}_{\lambda,C}$ is integral. Moreover, we see from (4.3) that $\mathfrak{m}_{\lambda,C}$ is a polynomial of the creation Heisenberg operators $\mathfrak{a}_{-i}(C),\ i > 0$ with rational coefficients.

4.2. Integrality.

Let α be an arbitrary class in $H^2(X)$. Fix a smooth irreducible curve C in the surface X. Recall from the previous subsection that the integral operator $\mathfrak{m}_{\lambda,C}$ is a polynomial of the creation Heisenberg operators $\mathfrak{a}_{-i}(C),\ i > 0$. This enable us to define $\mathfrak{m}_{\lambda,\alpha} \in \mathrm{End}(\mathbb{H}_X)$ by replacing the creation operators $\mathfrak{a}_{-i}(C)$ in $\mathfrak{m}_{\lambda,C}$ by the creation operators $\mathfrak{a}_{-i}(\alpha)$ correspondingly, and then define

$$[L^\lambda \alpha] = \mathfrak{m}_{\lambda,\alpha}|0\rangle. \tag{4.4}$$

Similarly, we can define the subspaces $\mathbb{H}_{n,\alpha}$ and $\mathbb{H}_\alpha$ of $\mathbb{H}_X$ as in (4.1).

Lemma 4.1. *Let $\alpha \in H^2(X)$ and $i > 0$. Then, we have*

$$\mathfrak{a}_{-i}(\alpha)[L^\lambda \alpha] = \sum_\mu a_{\lambda,\mu}[L^\mu \alpha] \tag{4.5}$$

where the summation is over partitions μ of $|\lambda|+i$, which are obtained as follows:

(i) *add i to a term in λ, say λ_k (possibly 0), and then*
(ii) *arrange it in descending order.*

The coefficient $a_{\lambda,\mu}$ is equal to the number of elements in $\{\ell \mid \mu_\ell = \lambda_k + i\}$.

Proof. Fix a smooth irreducible curve C in X. By the Theorem 9.14 in [Na2],

$$\mathfrak{a}_{-i}(C)[L^\lambda C] = \sum_\mu a_{\lambda,\mu}[L^\mu C]. \tag{4.6}$$

Define a linear map $\Psi_{C,\alpha} : \mathbb{H}_C \to \mathbb{H}_\alpha$ by sending the basis elements $\mathfrak{a}_{-\lambda}(C)|0\rangle$ of $\mathbb{H}_C$ to the elements $\mathfrak{a}_{-\lambda}(\alpha)|0\rangle$ in $\mathbb{H}_\alpha$. Note that the creation operators $\mathfrak{a}_{-j}(C)$ and $\mathfrak{a}_{-j}(\alpha)$ preserve $\mathbb{H}_C$ and $\mathbb{H}_\alpha$ respectively. Moreover, $\Psi_{C,\alpha} \circ \mathfrak{a}_{-j}(C) = \mathfrak{a}_{-j}(\alpha) \circ \Psi_{C,\alpha}$. It follows from the definition of $\mathfrak{m}_{\lambda,\alpha}$ that $\Psi_{C,\alpha} \circ \mathfrak{m}_{\lambda,C} = \mathfrak{m}_{\lambda,\alpha} \circ \Psi_{C,\alpha}$. Thus

$$\Psi_{C,\alpha}\big([L^\lambda C]\big) = \Psi_{C,\alpha} \circ \mathfrak{m}_{\lambda,C}|0\rangle = \mathfrak{m}_{\lambda,\alpha} \circ \Psi_{C,\alpha}|0\rangle = \mathfrak{m}_{\lambda,\alpha}|0\rangle = [L^\lambda \alpha]. \tag{4.7}$$

Now applying $\Psi_{C,\alpha}$ to both sides of (4.6), we obtain (4.5). □

Remark 4.2. It is a classical result (cf. [Mac, Na2]) that

$$p_i m_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu \tag{4.8}$$

where $a_{\lambda,\mu}$ is the same as defined in Lemma 4.1. This is compatible with (4.3).

Lemma 4.3. *Let $\alpha \in H^2(X)$. If $[L^\lambda \alpha]$ is integral for every λ, then so is $[L^\lambda(-\alpha)]$.*

Proof. Fix a smooth irreducible curve C in X, and let notations be the same as in the proof of Lemma 4.1. Recall from (4.3) that the linear isomorphism Φ_C sends m_λ and p_λ to $[L^\lambda C]$ and $\mathfrak{a}_{-\lambda}(C)|0\rangle$ respectively. Put

$$m_\lambda = \sum_{|\mu|=|\lambda|} d_\mu p_\mu \tag{4.9}$$

where $d_\mu \in \mathbb{Q}$. Applying $\Psi_{C,\alpha} \circ \Phi_C$ to both sides and using (4.7), we obtain

$$[L^\lambda \alpha] = \sum_{|\mu|=|\lambda|} d_\mu \mathfrak{a}_{-\mu}(\alpha)|0\rangle.$$

It follows from the definition of $[L^\lambda(-\alpha)]$ that

$$[L^\lambda(-\alpha)] = \sum_{|\mu|=|\lambda|} d_\mu (-1)^{\ell(\mu)} \mathfrak{a}_{-\mu}(\alpha)|0\rangle. \tag{4.10}$$

Next, it is well-known (see [Mac]) that there is an involution ω on Λ satisfying

$$\omega(p_\mu) = (-1)^{|\mu|-\ell(\mu)} p_\mu.$$

The so-called forgotten symmetric function associated to a partition μ is defined to be $f_\mu = \omega(m_\mu)$. The symmetric functions f_μ, $\mu \vdash n$ form another $\mathbb{Z}$-basis of Λ^n. In particular, f_λ is an integral linear combination of the monomial symmetric

functions $m_\mu,\ \mu \vdash |\lambda|$. Thus $\Psi_{C,\alpha} \circ \Phi_C(f_\lambda)$ is an integral linear combination of the classes $\Psi_{C,\alpha} \circ \Phi_C(m_\lambda) = [L^\mu \alpha],\ \mu \vdash |\lambda|$, and hence is an integral class. By (4.9),

$$\begin{aligned} \Psi_{C,\alpha} \circ \Phi_C(f_\lambda) &= \Psi_{C,\alpha} \circ \Phi_C \circ \omega \left(\sum_{|\mu|=|\lambda|} d_\mu p_\mu \right) \\ &= (-1)^{|\lambda|} \cdot \sum_{|\mu|=|\lambda|} d_\mu (-1)^{\ell(\mu)} \mathfrak{a}_{-\mu}(\alpha)|0\rangle. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we conclude that $[L^\lambda(-\alpha)]$ is integral as well. $\square$

Lemma 4.4. *Let $\alpha_1, \alpha_2 \in H^2(X)$, and $\alpha = \alpha_1 + \alpha_2$. Then,*

$$[L^\lambda \alpha] = \sum_{(\lambda^1,\lambda^2):\lambda^1 \cup \lambda^2 = \lambda} \mathfrak{m}_{\lambda^1,\alpha_1} \mathfrak{m}_{\lambda^2,\alpha_2}|0\rangle \tag{4.12}$$

where (λ^1, λ^2) stands for ordered pairs of partitions.

Proof. We start with some notations. For the partitions μ obtained from λ as in (4.5), we shall denote $\mu = \lambda \uparrow^i$. If we specify further that such a μ is obtained from adding i to a part of λ equal to j (here j is allowed to be 0), then we denote $\mu = \lambda \uparrow^i_j$. Given a partition λ, we denote by $m_k(\lambda)$ the multiplicity of the parts of λ equal to k. In these notations, the coefficient $a_{\lambda,\mu}$ in Lemma 4.1 for $\mu = \lambda \uparrow^i_j$ is simply equal to $m_{i+j}(\mu)$. Denote the right-hand-side of (4.12) by $R^\lambda(\alpha)$.

Claim. $\mathfrak{a}_{-i}(\alpha) R^\lambda(\alpha) = \sum_{\mu=\lambda\uparrow^i} a_{\lambda,\mu} R^\mu(\alpha)$.

Proof. The right-hand-side in the Claim is equal to

$$\sum_{\mu=\lambda\uparrow^i} \sum_{(\mu^1,\mu^2):\mu^1\cup\mu^2=\mu} a_{\lambda,\mu} \mathfrak{m}_{\mu^1,\alpha_1} \mathfrak{m}_{\mu^2,\alpha_2}|0\rangle. \tag{4.13}$$

On the other hand, by Lemma 4.1, the left-hand-side in the claim is

$$\begin{aligned} &\mathfrak{a}_{-i}(\alpha) R^\lambda(\alpha) \\ = &\sum_{(\lambda^1,\lambda^2):\lambda^1\cup\lambda^2=\lambda} \big(\mathfrak{m}_{\lambda^2,\alpha_2} \mathfrak{a}_{-i}(\alpha_1) \mathfrak{m}_{\lambda^1,\alpha_1}|0\rangle + \mathfrak{m}_{\lambda^1,\alpha_1} \mathfrak{a}_{-i}(\alpha_2) \mathfrak{m}_{\lambda^2,\alpha_2}|0\rangle \big) \\ = &\sum_{(\lambda^1,\lambda^2):\lambda^1\cup\lambda^2=\lambda} \sum_{\rho^1=\lambda^1\uparrow^i} a_{\lambda^1,\rho^1}\ \mathfrak{m}_{\rho^1,\alpha_1} \mathfrak{m}_{\lambda^2,\alpha_2}|0\rangle \qquad (4.14) \\ &+ \sum_{(\lambda^1,\lambda^2):\lambda^1\cup\lambda^2=\lambda} \sum_{\rho^2=\lambda^2\uparrow^i} a_{\lambda^2,\rho^2}\ \mathfrak{m}_{\lambda^1,\alpha_1} \mathfrak{m}_{\rho^2,\alpha_2}|0\rangle. \qquad (4.15) \end{aligned}$$

Note that the partitions $\rho^1 \cup \lambda^2$ and $\lambda^1 \cup \rho^2$ associated to $\lambda^1, \lambda^2, \rho^1$ and ρ^2 appearing above are of the form $\lambda \uparrow^i$. Thus the same type of terms appearing on both sides of the Claim. It remains to identify the coefficients of a given term.

Fix $\mu = \lambda \uparrow^i_j$ for some part j of λ, and fix μ^1, μ^2 such that $\mu^1 \cup \mu^2 = \mu$. From the above computation, the contributions to the term $\mathfrak{m}_{\mu^1,\alpha_1} \mathfrak{m}_{\mu^2,\alpha_2}|0\rangle$ in the left-hand-side of the Claim come from two places: the term in (4.14) for $\rho^1 = \mu^1, \lambda^2 = \mu^2$

whose coefficient is $m_{i+j}(\mu^1)$, and the term in (4.15) for $\lambda^1 = \mu^1, \rho^2 = \mu^2$ whose coefficient is $m_{i+j}(\mu^2)$. Therefore in view of (4.13), the coefficients of the term $\mathfrak{m}_{\mu^1,\alpha_1}\mathfrak{m}_{\mu^2,\alpha_2}|0\rangle$ in both sides of the Claim coincide thanks to

$$m_{i+j}(\mu^1) + m_{i+j}(\mu^2) = m_{i+j}(\mu^1 \cup \mu^2) = m_{i+j}(\mu) = a_{\lambda,\mu}. \tag{4.16}$$

This completes the proof of the above Claim. □

Next, we continue the proof of (4.12) by using induction on n and the reverse dominance ordering of partitions λ of n (we refer to p.7 of [Mac] for the definition of the dominance ordering). For $n = 1$, formula (4.12) is clear. Assume that formula (4.12) holds for all partitions of size less than n.

For $\lambda = (n)$, (4.12) holds since $\mathfrak{m}_{\lambda,\alpha} = \mathfrak{a}_{-n}(\alpha)$ and $[L^\lambda\alpha] = \mathfrak{a}_{-n}(\alpha)|0\rangle$.

For a general partition λ of n with a part equal to, say i, we denote by $\widetilde{\lambda}$ the partition obtained from λ with a part equal to i removed. Now replacing λ in (4.5) and the above Claim by $\widetilde{\lambda}$ respectively, we obtain

$$\mathfrak{a}_{-i}(\alpha)[L^{\widetilde{\lambda}}\alpha] \;=\; \sum_{\mu=\widetilde{\lambda}\uparrow^i} a_{\widetilde{\lambda},\mu}[L^\mu\alpha], \tag{4.17}$$

$$\mathfrak{a}_{-i}(\alpha)R^{\widetilde{\lambda}}(\alpha) \;=\; \sum_{\mu=\widetilde{\lambda}\uparrow^i} a_{\widetilde{\lambda},\mu}R^\mu(\alpha). \tag{4.18}$$

Note that λ appears among the above μ's as the maximum in the reverse dominance ordering. By induction hypothesis, the left-hand-sides of (4.17) and (4.18) coincide, and all the terms on the right-hand-sides of (4.17) and (4.18) involving μ not equal to λ coincide. Thus (4.12) follows since $a_{\widetilde{\lambda},\lambda} = m_i(\lambda) \neq 0$. □

Theorem 4.5. *For every divisor α on X and every partition λ, $[L^\lambda\alpha]$ is an integral class and $\mathfrak{m}_{\lambda,\alpha} \in \mathrm{End}(\mathbb{H}_X)$ is an integral operator.*

Proof. Every divisor α can be written as $C_1 - C_2$ for some very ample divisors C_1 and C_2. Represent C_1 and C_2 by smooth irreducible curves. Now we see from the discussions in Subsection 4.1, Lemma 4.3 and Lemma 4.4 that $[L^\lambda\alpha]$ is an integral class. By Proposition 3.5, $\mathfrak{m}_{\lambda,\alpha} \in \mathrm{End}(\mathbb{H}_X)$ is an integral operator. □

4.3. **Unimodularity.**

Our goal in this subsection is to prove the following.

Theorem 4.6. *Let $\alpha_1, \cdots, \alpha_k \in H^2(X)$ be linearly independent classes, and let $M_{\underline{\alpha}}$ be the intersection matrix of $\alpha_1, \cdots, \alpha_k$. Fix a positive integer n. Let $M_{n,\underline{\alpha}}$ be the intersection matrix of the classes in $H^{2n}(X^{[n]})$:*

$$\mathfrak{m}_{\lambda^1,\alpha_1} \cdots \mathfrak{m}_{\lambda^k,\alpha_k}|0\rangle, \quad |\lambda^1| + \ldots + |\lambda^k| = n. \tag{4.19}$$

If $\det M_{\underline{\alpha}} = \pm 1$*, then we have* $\det M_{n,\underline{\alpha}} = \pm 1$ *as well.*

First of all, we begin with some linear algebra preparations. Let V be a k-dimensional complex vector space with a symmetric bilinear form

$$(-,-) : V \times V \to \mathbb{C}.$$

Fix a linear basis $\mathbf{v} = \{v_1, \ldots, v_k\}$ of V, and let $M_{\mathbf{v}} = \big((v_i, v_j)\big)_{1 \le i,j \le k}$ be the matrix formed by the pairings (v_i, v_j). The following is elementary.

Lemma 4.7. *Let $\tilde{\mathbf{v}} = \{\tilde{v}_1, \ldots, \tilde{v}_k\}$ be another basis of V. Let T be the transition matrix from $\tilde{\mathbf{v}}^t$ to $\mathbf{v}^t$ (i.e., $\mathbf{v}^t = T\tilde{\mathbf{v}}^t$). Then, $M_{\mathbf{v}} = TM_{\tilde{\mathbf{v}}}T^t$.* □

For a given $n \ge 0$, the symmetric power $S^n(V)$ admits a monomial basis

$$\{v_{i_1} \cdots v_{i_n} |\ 1 \le i_1 \le \ldots \le i_n \le k\}. \tag{4.20}$$

Define a bilinear form, still denoted by $(-,-)$, on $S^n(V)$ by letting

$$(v_{i_1} \cdots v_{i_n}, v_{j_1} \cdots v_{j_n}) = \sum_{\sigma \in S_n} \prod_{a=1}^{n} (v_{i_a}, v_{j_{\sigma(a)}})$$

where S_n denotes the n-th symmetric group. Denote by $M_{n,\mathbf{v}}$ the matrix of the pairings of the monomial basis elements of $S^n(V)$ (so we have $M_{1,\mathbf{v}} = M_{\mathbf{v}}$).

Lemma 4.8. *For some constant $c(n,k)$, we have*

$$\det M_{n,\mathbf{v}} = c(n,k) \cdot (\det M_{\mathbf{v}})^{\binom{n+k-1}{k}}. \tag{4.21}$$

Proof. First we prove the formula under the assumption that

$$\det M_{\mathbf{v}_i} \ne 0 \quad \text{ for } 1 \le i \le k-1 \tag{4.22}$$

where $\mathbf{v}_i := \{v_1, \ldots, v_i\}$ for $1 \le i \le k$ (so $\mathbf{v}_k = \mathbf{v}$).

Use induction on k. When $k = 1$, our formula is clearly true.

Fix $k > 1$. Since $\det M_{\mathbf{v}_{k-1}} \ne 0$, there is a vector $\tilde{v}_k = e_1 v_1 + \ldots + e_{k-1} v_{k-1} + v_k$ such that $e_1, \ldots, e_{k-1} \in \mathbb{C}$ and $(v_1, \tilde{v}_k) = \ldots = (v_{k-1}, \tilde{v}_k) = 0$. Let

$$\tilde{\mathbf{v}} = \{v_1, \ldots, v_{k-1}, \tilde{v}_k\}.$$

Then the transition matrix from the basis $\tilde{\mathbf{v}}^t$ to the basis $\mathbf{v}^t$ is lower triangular with all diagonal entries being 1. By Lemma 4.7, we have

$$\det M_{\mathbf{v}} = \det M_{\tilde{\mathbf{v}}} = \det M_{\mathbf{v}_{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k). \tag{4.23}$$

Next, we apply Lemma 4.7 to $S^n(V)$ for the two monomial bases:

$$\{v_1^{i_1} \cdots v_{k-1}^{i_{k-1}} v_k^{i_k} |\ i_1 + \ldots + i_{k-1} + i_k = n\},$$
$$\{v_1^{i_1} \cdots v_{k-1}^{i_{k-1}} \tilde{v}_k^{i_k} |\ i_1 + \ldots + i_{k-1} + i_k = n\}.$$

Since the transition matrix from the second basis to the first basis is lower triangular with all diagonal entries being 1, we conclude that

$$\det M_{n,\mathbf{v}} = \det M_{n,\tilde{\mathbf{v}}}. \tag{4.24}$$

Now note that $(v_1^{i_1} \cdots v_{k-1}^{i_{k-1}} \tilde{v}_k^{i_k},\ v_1^{j_1} \cdots v_{k-1}^{j_{k-1}} \tilde{v}_k^{j_k})$ is equal to

$$(v_1^{i_1} \cdots v_{k-1}^{i_{k-1}},\ v_1^{j_1} \cdots v_{k-1}^{j_{k-1}}) \cdot \delta_{i_k, j_k} \cdot i_k! \cdot (\tilde{v}_k, \tilde{v}_k)^{i_k}.$$

It follows that $M_{n,\tilde{\mathbf{v}}} = \operatorname{diag}\big(\cdots,\ M_{n-m,\mathbf{v}_{k-1}} \cdot m! \cdot (\tilde{v}_k, \tilde{v}_k)^m,\ \cdots\big)$ where m runs from 0 to n. The matrix $M_{n-m,\mathbf{v}_{k-1}}$ has $\binom{n-m+k-2}{k-2}$ rows. So

$$\det M_{n,\tilde{\mathbf{v}}} = c_1(n,k) \cdot \prod_{m=0}^{n} \Big(\det M_{n-m,\mathbf{v}_{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k)^{m \cdot \binom{n-m+k-2}{k-2}} \Big)$$

for some constant $c_1(n,k)$. Applying induction to $\det M_{n-m,\mathbf{v}_{k-1}}$ yields

$$\begin{aligned} \det M_{n,\tilde{\mathbf{v}}} &= c(n,k) \cdot \prod_{m=0}^{n} \Big(\big(\det M_{\mathbf{v}_{k-1}} \big)^{\binom{n-m+k-2}{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k)^{m \cdot \binom{n-m+k-2}{k-2}} \Big) \\ &= c(n,k) \cdot \big(\det M_{\mathbf{v}_{k-1}} \big)^{\sum_{m=0}^{n} \binom{n-m+k-2}{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k)^{\sum_{m=0}^{n} m \cdot \binom{n-m+k-2}{k-2}} \\ &= c(n,k) \cdot \big(\det M_{\mathbf{v}_{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k) \big)^{\binom{n+k-1}{k}} \end{aligned} \tag{4.25}$$

for some constant $c(n,k)$, where we have used the combinatorial identities:

$$\sum_{m=0}^{n} \binom{n-m+k-2}{k-1} = \sum_{m=0}^{n} m \cdot \binom{n-m+k-2}{k-2} = \binom{n+k-1}{k} \tag{4.26}$$

Indeed, all the three terms in (4.26) compute the dimension of the space of degree-$(n-1)$ homogeneous polynomials in $(k+1)$-variables.

Combining (4.24), (4.25) and (4.23), we conclude that

$$\det M_{n,\mathbf{v}} = c(n,k) \cdot \big(\det M_{\mathbf{v}_{k-1}} \cdot (\tilde{v}_k, \tilde{v}_k) \big)^{\binom{n+k-1}{k}} = c(n,k) \cdot \big(\det M_{\mathbf{v}} \big)^{\binom{n+k-1}{k}}.$$

Finally, we come to the general case. Assume that the bilinear form $(-,-)$ on V is not identically zero (otherwise the lemma trivially holds). Set $z_{ij} = (v_i, v_j)$ for $1 \le i \le j \le k$. Then both sides of (4.21) are easily seen to be polynomials in the variables $z_{ij},\ 1 \le i \le j \le k$. The proof above under the assumption (4.22) implies that the (polynomial) identity (4.21) holds for a Zariski open subset $(z_{ij})_{1 \le i \le j \le k}$ of $\mathbb{C}^{k(k+1)/2}$, and thus it holds for an arbitrary $(z_{ij})_{1 \le i \le j \le k}$. □

For $r \ge 1$, we denote by $V[r]$ a copy of V with bilinear form

$$(-,-)_r = (-1)^{r-1} r \cdot (-,-). \tag{4.27}$$

We shall denote by $(-,-)_r$ as well the induced bilinear form on the symmetric power $S^*(V[r])$. In particular, $V[1] = V$ with $(-,-)_1 = (-,-)$.

For a partition $\mu = (r^{m_r})_{r \ge 1}$, we form the vector space $S^\mu V := \bigotimes_{r \ge 1} S^{m_r}(V[r])$ with a bilinear form $(-,-)_\mu$ given by $(\otimes_r u_{I_r}, \otimes_r v_{J_r})_\mu = \prod_r (u_{I_r}, v_{J_r})_r$ for $u_{I_r}, v_{J_r} \in S^{m_r}(V[r])$. Denote by $M_{\mu,\mathbf{v}}$ the matrix of the pairings of the induced monomial basis for $S^\mu V$ (see (4.20) for the monomial basis of $S^n(V)$).

Lemma 4.9. *For some constant $c(\mu,k)$ and some integer $d(\mu,k) \ge 1$, we have*

$$\det M_{\mu,\mathbf{v}} = c(\mu,k) \cdot \big(\det M_{\mathbf{v}} \big)^{d(\mu,k)}.$$

Proof. Follows immediately from our definitions and Lemma 4.8. □

Now let $\alpha_1, \ldots, \alpha_k \in H^2(X)$ be from Theorem 4.6. Let $V = \mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_k$ with the pairing induced from the one on $H^*(X)$. Let $\mathbb{H}_{n,V} \subset H^*(X^{[n]})$ be the $\mathbb{C}$-linear span of the classes (4.19). Then, $\mathbb{H}_{n,V}$ has two linear bases:

$$\begin{aligned} \mathfrak{m}_{n,\underline{\alpha}} &:= \{\mathfrak{m}_{\lambda^1,\alpha_1} \cdots \mathfrak{m}_{\lambda^k,\alpha_k}|0\rangle\}, \qquad (4.28)\\ \mathfrak{a}_{n,\underline{\alpha}} &:= \{\mathfrak{a}_{-\lambda^1}(\alpha_1) \cdots \mathfrak{a}_{-\lambda^k}(\alpha_k)|0\rangle\} \qquad (4.29)\end{aligned}$$

where $\lambda^1, \ldots, \lambda^k$ are partitions satisfying $|\lambda^1| + \cdots + |\lambda^k| = n$.

Take an orthogonal basis $\{\beta_1, \cdots, \beta_k\}$ of V with $(\beta_i, \beta_i) = \pm 1$ for every i. We define similarly bases $\mathfrak{m}_{n,\underline{\beta}}$ and $\mathfrak{a}_{n,\underline{\beta}}$ for $\mathbb{H}_{n,V}$. Observe that the transition matrix B from the basis $\mathfrak{m}_{n,\underline{\alpha}}$ to $\mathfrak{a}_{n,\underline{\alpha}}$ is independent of the basis $\{\alpha_1, \cdots, \alpha_k\}$ of $H^2(X)$, that is, it is the same as the transition matrix from the basis $\mathfrak{m}_{n,\underline{\beta}}$ to $\mathfrak{a}_{n,\underline{\beta}}$.

Denote by A the transition matrix from the basis $\mathfrak{a}_{n,\underline{\beta}}$ to $\mathfrak{a}_{n,\underline{\alpha}}$. Denote by $M_{n,\underline{\beta}}$ the intersection matrix of the basis $\mathfrak{m}_{n,\underline{\beta}}$. Since the transition matrix from $\mathfrak{m}_{n,\underline{\beta}}$ to $\mathfrak{m}_{n,\underline{\alpha}}$ is $B^{-1}AB$, we obtain from Lemma 4.7 the following.

Lemma 4.10. $M_{n,\underline{\alpha}} = (B^{-1}AB)\ M_{n,\underline{\beta}}\ (B^{-1}AB)^t$. □

Next, we prove a special case of Theorem 4.6 for the basis $\{\beta_1, \cdots, \beta_k\}$.

Lemma 4.11. $\det M_{n,\underline{\beta}} = \pm 1$.

Proof. As complex vector spaces, we have a linear isomorphism:

$$\mathbb{H}_{n,V} \cong \bigoplus_{\underline{n}} \bigotimes_{i=1}^{k} \mathbb{H}_{n_i,\mathbb{C}\beta_i} \qquad (4.30)$$

where the sum is over $\underline{n} = (n_1, \cdots, n_k) \in (\mathbb{Z}_{\geq 0})^k$ such that $n_1 + \ldots + n_k = n$. Denote by M_{n_i,β_i} the matrix of pairings among the basis $\big\{\mathfrak{m}_{\lambda^i,\beta_i}|0\rangle \mid \lambda^i \vdash n_i\big\}$ for $\mathbb{H}_{n_i,\mathbb{C}\beta_i}$. For given $1 \leq i \leq k$ and λ^i, the operator $\mathfrak{m}_{\lambda^i,\beta_i}$ is a linear combination of the operators $\mathfrak{a}_{-\mu^i}(\beta_i)$, $\mu^i \vdash |\lambda^i|$. Since the classes $\beta_1, \cdots, \beta_k$ are orthogonal, we see from (2.5) that $\big(\mathfrak{m}_{\lambda^1,\beta_1} \cdots \mathfrak{m}_{\lambda^k,\beta_k}|0\rangle, \mathfrak{m}_{\mu^1,\beta_1} \cdots \mathfrak{m}_{\mu^k,\beta_k}|0\rangle\big)$ is equal to

$$\prod_{i=1}^{k} \big(\delta_{|\lambda^i|,|\mu^i|}\ \cdot\ \big(\mathfrak{m}_{\lambda^i,\beta_i}|0\rangle, \mathfrak{m}_{\mu^i,\beta_i}|0\rangle\big)\big)\,.$$

This orthogonality together with (4.30) implies by standard linear algebra that

$$\det M_{n,\underline{\beta}} = \prod_{\underline{n}} \prod_{i=1}^{k} (\det M_{n_i,\beta_i})^{\prod_{1\leq j\leq k, j\neq i} \dim \mathbb{H}_{n_j,\mathbb{C}\beta_j}}.$$

Thus it suffices to prove our lemma for $k = 1$.

Put $\beta = \beta_1$ and $V = \mathbb{C}\beta$. By (4.28) and (4.29), $\mathbb{H}_{n,V}$ has two linear bases:

$$\mathfrak{m}_{n,\underline{\beta}} = \big\{\mathfrak{m}_{\lambda,\beta}|0\rangle \mid \lambda \vdash n\big\}, \qquad \mathfrak{a}_{n,\underline{\beta}} = \big\{\mathfrak{a}_{-\lambda}(\beta)|0\rangle \mid \lambda \vdash n\big\}.$$

Since $(\beta, \beta) = \pm 1$, we obtain by (2.4) and (2.5) that

$$\big(\mathfrak{a}_{-\lambda}(\beta)|0\rangle, \mathfrak{a}_{-\mu}(\beta)|0\rangle\big) = \pm\delta_{\lambda,\mu} \cdot \mathfrak{z}_\lambda \cdot (\beta, \beta)^{\ell(\lambda)} = \pm\delta_{\lambda,\mu} \cdot \mathfrak{z}_\lambda. \qquad (4.31)$$

Recall that the transition matrix from the basis $\mathfrak{a}_{n,\underline{\beta}}$ to $\mathfrak{m}_{n,\underline{\beta}}$ is B^{-1}. So we conclude from Lemma 4.7 and (4.31) that $M_{n,\underline{\beta}} = B^{-1} \cdot \operatorname{diag}(\cdots, \pm\mathfrak{z}_\lambda, \cdots) \cdot (B^{-1})^t$ where λ runs over all partitions of n. Hence we have

$$\det M_{n,\underline{\beta}} \;=\; \pm(\det B)^{-2} \cdot \prod_{\lambda \vdash n} \mathfrak{z}_\lambda \tag{4.32}$$

By (4.3), Lemma 4.1, and (4.8), we see that the transition matrix from the basis $\{p_\lambda |\ \lambda \vdash n\}$ of $\Lambda^n_{\mathbb{Q}} \subset \Lambda_{\mathbb{Q}}$ to the basis $\{m_\lambda |\ \lambda \vdash n\}$ is B^{-1} as well. Introduce the standard bilinear form $(-,-)$ on $\Lambda^n_{\mathbb{Q}}$ (cf. [Mac]) by letting

$$(p_\lambda, p_\mu) = \delta_{\lambda,\mu} \cdot \mathfrak{z}_\lambda. \tag{4.33}$$

The matrix M_n formed by the pairings of the monomial symmetric functions $m_\lambda,\ \lambda \vdash n$ is unimodular, and thus $\det M_n = \pm 1$. By Lemma 4.7 and (4.33),

$$M_n \;=\; B^{-1} \cdot \operatorname{diag}(\cdots, \mathfrak{z}_\lambda, \cdots) \cdot (B^{-1})^t.$$

where λ runs over all partitions of n. It follows immediately that

$$(\det B)^{-2} \cdot \prod_{\lambda \vdash n} \mathfrak{z}_\lambda = \det M_n = \pm 1. \tag{4.34}$$

Combining this with (4.32), we finally obtain $\det M_{n,\underline{\beta}} = \pm 1$. $\square$

For $\mu \vdash n$, let $\mathbb{H}_{\mu,\underline{\alpha}} \subset \mathbb{H}_{n,V}$ be the span of $\mathfrak{a}_{-\mu^1}(\alpha_1) \cdots \mathfrak{a}_{-\mu^k}(\alpha_k)|0\rangle$ where $\mu^1, \ldots, \mu^k$ are partitions such that $\mu^1 \cup \cdots \cup \mu^k = \mu$. Note that $\mathbb{H}_{\mu,\underline{\alpha}} \subset \mathbb{H}_{n,V} \subset H^{2n}(X^{[n]})$. So $\mathbb{H}_{\mu,\underline{\alpha}}$ carries a pairing induced from the one on $H^{2n}(X^{[n]})$.

Lemma 4.12. *For every partition μ, we have an isometry $\mathbb{H}_{\mu,\underline{\alpha}} \cong S^\mu(V)$.*

Proof. Follows from the definitions, (4.27) and the commutation relation (2.5). $\square$

Lemma 4.13. $(\det A)^2 = \pm 1$.

Proof. By the Heisenberg algebra commutation relation (2.5), $\mathfrak{a}_{-\mu^1}(*) \cdots \mathfrak{a}_{-\mu^k}(*)$ is orthogonal to $\mathfrak{a}_{-\nu^1}(*) \cdots \mathfrak{a}_{-\nu^\ell}(*)|0\rangle$ unless $\mu^1 \cup \cdots \cup \mu^k = \nu^1 \cup \cdots \cup \nu^\ell$. Here $*$'s denote unspecified classes in $H^2(X)$. Thus, we have an orthogonal direct sum

$$\mathbb{H}_{n,V} = \bigoplus_{\mu \vdash n} \mathbb{H}_{\mu,\underline{\alpha}}.$$

By Lemma 4.12, the intersection matrix $\widetilde{M}_{n,\underline{\alpha}}$ of the basis $\mathfrak{a}_{n,\underline{\alpha}}$ for $\mathbb{H}_{n,V}$ is given by the diagonal block matrix whose diagonal consists of $M_{\mu,\underline{\alpha}}$, $\mu \vdash n$. So

$$\det \widetilde{M}_{n,\underline{\alpha}} = \prod_{\mu \vdash n} \det M_{\mu,\underline{\alpha}}. \tag{4.35}$$

Similarly, by repeating the above with the α's replaced by the β's, we obtain

$$\det \widetilde{M}_{n,\underline{\beta}} = \prod_{\mu \vdash n} \det M_{\mu,\underline{\beta}}. \tag{4.36}$$

By assumptions, $\det M_{\underline{\alpha}} = \pm 1$ and $\det M_{\underline{\beta}} = \pm 1$. Thus by Lemma 4.9, we have $\det M_{\mu,\underline{\alpha}} = \pm \det M_{\mu,\underline{\beta}}$ for every μ. Therefore, we see from (4.35) and (4.36) that

$$\det \widetilde{M}_{n,\underline{\alpha}} = \pm \det \widetilde{M}_{n,\underline{\beta}}. \tag{4.37}$$

By Lemma 4.7 and the definition of the matrix A, we have $\widetilde{M}_{n,\underline{\alpha}} = A\,\widetilde{M}_{n,\underline{\beta}} A^t$. Combining this with (4.37) yields $(\det A)^2 = \pm 1$. $\square$

Finally, Theorem 4.6 follows from Lemmas 4.10, 4.11, and 4.13.

5. Integral bases for the cohomology of Hilbert schemes

5.1. Structure of integral bases for $H^*(X^{[n]})$.

Definition 5.1. For $n \ge 0$, define $\mathbb{H}'_{n,X}$ to be the linear subspace of $\mathbb{H}_{n,X} := H^*(X^{[n]})$ spanned by all Heisenberg monomial classes involving only the creation operators $\mathfrak{a}_{-j}(\alpha)$ where $j > 0$ and $\alpha \in H^1(X) \oplus H^2(X) \oplus H^3(X)$.

Parallel to (3.1), every class $A \in H^*(X^{[n]})$ can be written as

$$A = \sum_{\lambda,\mu} \frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)(A_{\lambda,\mu}) \tag{5.1}$$

where $A_{\lambda,\mu} \in \mathbb{H}'_{n-|\lambda|-|\mu|,X}$. The following is an analogue of Lemma 3.4.

Lemma 5.2. *Let $A \in H^*(X^{[n]})$ be expressed as in (5.1). Then,*

(i) *the classes $A_{\lambda,\mu}$ are uniquely determined by A;*
(ii) *A is integral if and only if all the classes $A_{\lambda,\mu}$ are integral.*

Proof. Our argument is similar to the one used in the proof of Lemma 3.4. First of all, the operator $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)$ are integral by Lemma 3.6. Since $\mathfrak{a}_{-\mu}(x)$ is also integral, we see that the class A is integral if all the classes $A_{\lambda,\mu}$ are integral. So it remains to prove (i) and the "only if" part of (ii).

Next, let n_0 be the maximal integer such that $A_{\lambda,\mu} \ne 0$ for some partitions λ, μ with $|\lambda| + |\mu| = n_0$. If λ and μ are partitions with $|\lambda| + |\mu| = n_0$, then applying the adjoint operator $\big(1/\mathfrak{z}_\mu \cdot \mathfrak{a}_{-\mu}(1)\mathfrak{a}_{-\lambda}(x)\big)^\dagger$ to both sides of (5.1) yields:

$$\big(1/\mathfrak{z}_\mu \cdot \mathfrak{a}_{-\mu}(1)\mathfrak{a}_{-\lambda}(x)\big)^\dagger(A) = (-1)^{|\lambda|-\ell(\lambda)+|\mu|-\ell(\mu)} \cdot A_{\lambda,\mu}.$$

So the class $A_{\lambda,\mu}$ is uniquely determined by A. Moreover, by Lemma 3.2 (i), if A is an integral class, then so is the class $A_{\lambda,\mu}$.

Finally, repeating the above process to the class

$$\begin{aligned} A' &:= A - \sum_{|\lambda|+|\mu|=n_0} \frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)(A_{\lambda,\mu}) \\ &= \sum_{|\lambda|+|\mu|<n_0} \frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)(A_{\lambda,\mu}), \end{aligned}$$

we conclude that (i) and the "only if" part of (ii) hold for all the classes $A_{\lambda,\mu}$. $\square$

Proposition 5.3. *Assume that for each fixed $k \geq 0$, the cohomology classes $B_{k,i}, i \in \mathcal{I}_k$ form an integral basis of $\mathbb{H}'_{k,X}$, where $\mathcal{I}_k$ is an index set depending on k. Then, an integral basis of $H^*(X^{[n]})$ consists of the classes:*

$$\frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)B_{k,i}, \qquad |\lambda| + |\mu| + k = n,\ i \in \mathcal{I}_k. \tag{5.2}$$

Proof. Since $B_{k,i}$ are integral classes and the operators $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)$ are integral, the cohomology classes (5.2) are integral. By Lemma 5.2 (ii), every integral class in $H^*(X^{[n]})$ is an integral linear combination of the classes (5.2). Since the dimension of $H^*(X^{[n]})$ is equal to the number of elements in (5.2), we conclude that an integral basis of $H^*(X^{[n]})$ is given by the classes (5.2). □

This proposition enables us to write down an integral basis of $H^*(X^{[n]})$ whenever an integral basis of the subspace $\mathbb{H}'_{k,X}$ is known for every $k \leq n$.

5.2. Surfaces X with $H^1(X; \mathcal{O}_X) = H^2(X; \mathcal{O}_X) = 0$.

Theorem 5.4. *Let X be a projective surface with $H^1(X; \mathcal{O}_X) = H^2(X; \mathcal{O}_X) = 0$. Let $\alpha_1, \cdots, \alpha_k$ be an integral basis of $H^2(X)$. Then the following classes*

$$\frac{1}{\mathfrak{z}_\lambda} \cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)\mathfrak{m}_{\nu^1,\alpha_1} \cdots \mathfrak{m}_{\nu^k,\alpha_k}|0\rangle, \quad |\lambda| + |\mu| + \sum_{i=1}^{k} |\nu^i| = n \tag{5.3}$$

are integral, and furthermore, they form an integral basis for $H^*(X^{[n]}; \mathbb{Z})/\mathrm{Tor}$.

Proof. By Proposition 5.3, it remains to show that for each fixed $\ell \geq 0$, an integral basis of $\mathbb{H}'_{\ell,X}$ is given by the integral cohomology classes:

$$\mathfrak{m}_{\nu^1,\alpha_1} \cdots \mathfrak{m}_{\nu^k,\alpha_k}|0\rangle, \quad \sum_{i=1}^{k} |\nu^i| = \ell. \tag{5.4}$$

Since $H^1(X; \mathcal{O}_X) = H^2(X; \mathcal{O}_X) = 0$, we have $H^2(X; \mathbb{Z}) \cong \mathrm{Pic}(X)$ and $H^1(X) = H^3(X) = 0$. Hence the integral classes $\alpha_1, \cdots, \alpha_k$ are divisors. By Theorem 4.5, all the classes in (5.4) are integral. By our assumption, the intersection matrix of the classes $\alpha_1, \cdots, \alpha_k$ is unimodular. It follows from Theorem 4.6 that the intersection matrix of the classes (5.4) is unimodular as well. Since $H^1(X) = H^3(X) = 0$, the dimension of the space $\mathbb{H}'_{\ell,X}$ is precisely equal to the number of classes in (5.4). Therefore, the classes (5.4) form an integral basis of $\mathbb{H}'_{\ell,X}$. □

We conjecture that the cohomology class $[L^\lambda \alpha]$ is integral whenever $\alpha \in H^2(X)$ is an integral class. If this conjecture is true, then the statement in Theorem 5.4 will be valid for every projective surface X with vanishing odd cohomology.

Remark 5.5. Let $C_1, \cdots, C_k$ be smooth irreducible curves in X such that any two of them intersect transversely and no three of them intersect. Let $\nu^1, \cdots, \nu^k$ be partitions with $|\nu^1| + \cdots + |\nu^k| = \ell$. Then, $\mathfrak{m}_{\nu^1,C_1} \cdots \mathfrak{m}_{\nu^k,C_k}|0\rangle \in H^{2\ell}(X^{[\ell]})$ is the fundamental class of the closure of the following subvariety in $X^{[\ell]}$:

$$\{\xi_1 + \ldots + \xi_k |\ \xi_i \in L^{\nu_i} C_i \text{ for every } i,\ \mathrm{Supp}(\xi_i) \cap \mathrm{Supp}(\xi_j) = \emptyset \text{ for } i \neq j\}.$$

This follows from an induction on k and $\ell(\nu_k)$, together with (4.5) and an argument similar to the proof of the Theorem 9.14 in [Na2].

Remark 5.6. As noted by the referee, since the monodromy operators on the cohomology of a surface preserve the Heisenberg operators on the integral cohomology of Hilbert schemes, the operator $\mathfrak{m}_{\lambda,\alpha}$ for a *non-algebraic* class $\alpha \in H^2(X;\mathbb{Z})/\mathrm{Tor}$ is integral if there exists a monodromy sending α to a divisor. For instance, when X is a K3 surface, it is known that for any class $\alpha \in H^2(X;\mathbb{Z})$, there exists a monodromy sending α to a divisor. Hence the conclusions in Theorem 5.4 hold when X is a K3 surface.

5.3. **An algebraic model.**

The purpose of this subsection is to formalize the constructions in the previous sections in a purely algebraic way.

Given a finite-dimensional graded Frobenius algebra A over $\mathbb{Q}$, one can construct a Fock space $\mathbb{H}_A$ of Heisenberg algebra generated by $\mathfrak{a}_n(\alpha), n \in \mathbb{Z}, \alpha \in A$. For the sake of simplicity, we assume that A is evenly graded, i.e., $A = \oplus_{i=0}^r A_{2i}$, with A_0 and A_{2r} being 1-dimensional. Then, $\mathbb{H}_A$ is bigraded, and its n-th component $A^{[n]}$ is still graded; in particular, $A^{[1]} = A$. Furthermore, as graded vector spaces

$$\mathbb{H}_A \cong \mathbb{H}_{A_{\mathrm{mid}}} \otimes \mathbb{H}_{A_0 \oplus A_{2r}}$$

where $A_{\mathrm{mid}} = \oplus_{i=1}^{r-1} A_{2i}$ inherits a non-degenerate bilinear form by restriction from A, $\mathbb{H}_{A_{\mathrm{mid}}}$ and $\mathbb{H}_{A_0\oplus A_{2r}}$ are the Fock spaces associated to A_{mid} and $A_0 \oplus A_{2r}$ respectively. We denote by $A_{\mathrm{mid}}^{[n]}$ the n-th component of $\mathbb{H}_{A_{\mathrm{mid}}}$. Given an integral lattice L_A in A_{mid} and an integral basis $\alpha_1, \ldots, \alpha_k$ of it, we can define the operators $\mathfrak{a}_{-\mu}(\alpha)$ associated to a partition μ as before, and then define the operators $\mathfrak{m}_{\lambda,\alpha}$ from the operators $\mathfrak{a}_{-\mu}(\alpha)$ by declaring that the transition matrix is the same as the (universal) one between the monomial symmetric functions and the power-sum symmetric functions. Then we introduce the elements $m_\lambda := \mathfrak{m}_{\lambda^1,\alpha_1} \ldots \mathfrak{m}_{\lambda^k,\alpha_k}|0\rangle \in \mathbb{H}_A$ associated to k-tuple partitions $\lambda = (\lambda^1, \cdots, \lambda^k)$. We denote by $L_A^{[n]}$ the lattice of $A_{\mathrm{mid}}^{[n]}$ which is the $\mathbb{Z}$-span of the elements m_λ with $|\lambda^1| + \cdots |\lambda^k| = n$. Then by construction, the operators $\mathfrak{m}_{\lambda^i,\alpha_i}, 1 \le i \le k$, are "integral", i.e. they preserve $\oplus_{n\ge 0} L_A^{[n]}$. Furthermore, the counterparts of Lemmas 4.3 and 4.4 hold in the current setup and they readily imply that the lattice $L_A^{[n]}$ is independent of the choice of the integral basis $\alpha_1, \ldots, \alpha_k$ of L_A. If the determinant of the intersection matrix of the lattice L_A is ± 1, then the same arguments as before imply that the determinant of the intersection matrix of $L_A^{[n]}$ for each n is ± 1. However, it is not clear to us whether or not the pairing on $L_A^{[n]}$ induced from the integral pairing on L_A is integral.

Remark 5.7. Assume further that we have an orthogonal direct sum $A_{\mathrm{mid}} = B \oplus C$. Choose lattices L_A, L_B and L_C of A_{mid}, B and C respectively such that $L_A = L_B \oplus L_C$. As before, we can define the Fock spaces $\mathbb{H}_B$ and $\mathbb{H}_C$, their n-th components $B^{[n]}$ and $C^{[n]}$, and the lattices $L_B^{[n]}$ and $L_C^{[n]}$ for each n. Clearly, $\mathbb{H}_{A_{\mathrm{mid}}} \cong \mathbb{H}_B \otimes \mathbb{H}_C$. Furthermore, by constructions the lattice structures are compatible with the tensor

product decomposition of Fock spaces:

$$\left(\oplus_{n\geq 0}L_A^{[n]}\right) \cong \left(\oplus_{n\geq 0}L_B^{[n]}\right) \bigotimes \left(\oplus_{n\geq 0}L_C^{[n]}\right). \tag{5.5}$$

On the other hand, take the lattice $M = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot x$, where 1 is the unit of the Frobenius algebra A and $x \in A_{2r}$ has trace one. We can define an integral lattice $M^{[n]}$ in the n-th component of the Fock space $\mathbb{H}_{A_0\oplus A_{2r}}$ which is the $\mathbb{Z}$-span of $\frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)$, where λ, μ are partitions such that $|\lambda| + |\mu| = n$. Now, $\big(\oplus_{n\geq 0} L^{[n]}\big) \otimes \big(\oplus_{m\geq 0} M^{[m]}\big)$ is a lattice of the Fock space $\mathbb{H}_A = \mathbb{H}_{A_{\mathrm{mid}}} \otimes \mathbb{H}_{A_0\oplus A_{2r}}$. In other words, $\oplus_{i=0}^n L^{[n-i]} \otimes M^{[i]}$ can be identified with a lattice in $A^{[n]}$.

By taking $A = H^*(X)$, the Heisenberg algebra attains a geometric meaning by Nakajima's construction [Na2]. In this way, our Theorem 5.4 can be restated that for a projective surface X with the given conditions, the lattice $\oplus_{i=0}^n L^{[n-i]} \otimes M^{[i]}$ is identified with $H^*(X^{[n]};\mathbb{Z})/\mathrm{Tor}$. Our conjecture in the paragraph following Theorem 5.4 can be further reformulated by saying that $\oplus_{i=0}^n L^{[n-i]} \otimes M^{[i]}$ can be identified with $H^*(X^{[n]};\mathbb{Z})/\mathrm{Tor}$ for every simply-connected surface X.

5.4. **Blown-up surfaces.**

Let $\widetilde{X}$ be the blown-up of X at one point, and E be the exceptional curve. In this subsection, we study a relation between integral bases of $H^*(\widetilde{X}^{[n]})$ and $H^*(X^{[n]})$. Regard $H^*(X;\mathbb{Z})/\mathrm{Tor}$ as a sublattice of $H^*(\widetilde{X};\mathbb{Z})/\mathrm{Tor}$, and $H^*(X)$ as a subspace of $H^*(\widetilde{X})$. Fix an integral basis $\{\alpha_1,\ldots,\alpha_k\}$ of $\big(H^1(X)\oplus H^2(X)\oplus H^3(X)\big)/\mathrm{Tor}$. Then, $\{\alpha_1,\ldots,\alpha_k, E\}$ is an integral basis of $\big(H^1(\widetilde{X})\oplus H^2(\widetilde{X})\oplus H^3(\widetilde{X})\big)/\mathrm{Tor}$.

Define $\mathbb{H}''_{n,\widetilde{X}}$ to be the linear subspace of $\mathbb{H}'_{n,\widetilde{X}} \subset \mathbb{H}_{n,\widetilde{X}} = H^*(\widetilde{X}^{[n]})$ spanned by all Heisenberg monomial classes involving only the creation operators $\mathfrak{a}_{-j}(\alpha)$ with

$$j > 0, \qquad \alpha \in \bigoplus_{i=1}^3 H^i(X) \subset \bigoplus_{i=1}^3 H^i(\widetilde{X}). \tag{5.6}$$

Let $\widetilde{A} \in H^*(\widetilde{X}^{[n]})$. As in (5.1), $\widetilde{A}$ can be written as

$$\widetilde{A} = \sum_{\lambda,\mu,\nu} \frac{1}{\mathfrak{z}_\lambda}\cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)\mathfrak{a}_{-\nu}(E)(\widetilde{B}_{\lambda,\mu,\nu}) \tag{5.7}$$

where $\widetilde{B}_{\lambda,\mu,\nu} \in \mathbb{H}''_{n-|\lambda|-|\mu|-|\nu|,\widetilde{X}}$ for partitions λ, μ and ν. Therefore, we see from (4.3) and (5.7) that $\widetilde{A}$ can be further rewritten as

$$\widetilde{A} = \sum_{\lambda,\mu,\nu} \frac{1}{\mathfrak{z}_\lambda}\cdot \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(x)\mathfrak{m}_{\nu,E}(\widetilde{A}_{\lambda,\mu,\nu}) \tag{5.8}$$

where $\widetilde{A}_{\lambda,\mu,\nu} \in \mathbb{H}''_{n-|\lambda|-|\mu|-|\nu|,\widetilde{X}}$. The following is similar to Lemma 5.2.

Proposition 5.8. *Let $\widetilde{A} \in H^*(\widetilde{X}^{[n]})$ be expressed as in (5.8). Then,*

(i) *the classes $\widetilde{A}_{\lambda,\mu,\nu}$ are uniquely determined by $\widetilde{A}$;*

(ii) *$\widetilde{A}$ is integral if and only if all the classes $\widetilde{A}_{\lambda,\mu,\nu}$ are integral.*

Proof. We follow an argument similar to the one used in the proof of Lemma 5.2. First of all, since $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1), \mathfrak{a}_{-\mu}(x)$ and $\mathfrak{m}_{\nu,E}$ are integral, $\widetilde{A}$ is integral if all the classes $\widetilde{A}_{\lambda,\mu,\nu}$ are integral. So it remains to prove (i) and the "only if" part of (ii).

Next, in view of Lemma 5.2, it suffices to show that if $\widetilde{A} = \sum_\nu \mathfrak{m}_{\nu,E}(\widetilde{A}_\nu)$ where $\widetilde{A}_\nu \in \mathbb{H}''_{n-|\nu|,\widetilde{X}}$ for partitions ν, then we have

(i′) the classes $\widetilde{A}_\nu$ are uniquely determined by $\widetilde{A}$;
(ii′) the classes $\widetilde{A}_\nu$ are integral when $\widetilde{A}$ is integral.

Let n_0 be the maximal integer such that $\widetilde{A}_\nu \neq 0$ for some $\nu \vdash n_0$. Consider the action of all the adjoint operators $(\mathfrak{m}_{\rho,E})^\dagger$, $\rho \vdash n_0$ on $\widetilde{A}$:

$$(\mathfrak{m}_{\rho,E})^\dagger(\widetilde{A}) = \sum_\nu (\mathfrak{m}_{\rho,E})^\dagger \mathfrak{m}_{\nu,E}(\widetilde{A}_\nu), \qquad \rho \vdash n_0. \tag{5.9}$$

Recall from (4.3) that $\mathfrak{m}_{\rho,E}$ is a polynomial of the creation operators $\mathfrak{a}_{-i}(E),\ i > 0$. So $(\mathfrak{m}_{\rho,E})^\dagger$ is a polynomial of the operators $\mathfrak{a}_i(E),\ i > 0$. By the definition of $\mathbb{H}''_{k,\widetilde{X}}$ and the fact that $(E, \alpha) = 0$ for $\alpha \in H^*(X)$, if we write $\widetilde{A}_\nu = \mathfrak{a}_{\widetilde{A}_\nu}|0\rangle$ as in Proposition 3.5, then $\big[(\mathfrak{m}_{\rho,E})^\dagger, \mathfrak{a}_{\widetilde{A}_\nu}\big] = 0$. Thus, for $\rho \vdash n_0$, we see from (5.9) that

$$\begin{aligned}(\mathfrak{m}_{\rho,E})^\dagger(\widetilde{A}) &= \sum_\nu \mathfrak{a}_{\widetilde{A}_\nu}(\mathfrak{m}_{\rho,E})^\dagger \mathfrak{m}_{\nu,E}|0\rangle = \sum_{|\nu|=n_0} \mathfrak{a}_{\widetilde{A}_\nu}(\mathfrak{m}_{\rho,E})^\dagger \mathfrak{m}_{\nu,E}|0\rangle \\ &= \sum_{|\nu|=n_0} \big(\mathfrak{m}_{\rho,E}|0\rangle, \mathfrak{m}_{\nu,E}|0\rangle\big) \cdot \mathfrak{a}_{\widetilde{A}_\nu}|0\rangle = \sum_{|\nu|=n_0} \big(\mathfrak{m}_{\rho,E}|0\rangle, \mathfrak{m}_{\nu,E}|0\rangle\big) \cdot \widetilde{A}_\nu.\end{aligned} \tag{5.10}$$

Applying Lemma 4.11 to $\underline{\beta} = \{E\}$, we see that the intersection matrix of the integral classes $\mathfrak{m}_{\rho,E}|0\rangle, \rho \vdash n_0$ is unimodular. Therefore by (5.10), the classes $\widetilde{A}_\nu, \nu \vdash n_0$ are integral linear combinations of the classes $(\mathfrak{m}_{\rho,E})^\dagger(\widetilde{A}), \rho \vdash n_0$. So the classes $\widetilde{A}_\nu, \nu \vdash n_0$ are uniquely determined by $\widetilde{A}$. Moreover, in view of Lemma 3.2 (i), if the class $\widetilde{A}$ is integral, then all the classes $(\mathfrak{m}_{\rho,E})^\dagger(\widetilde{A}), \rho \vdash n_0$ are integral. Hence all the classes $\widetilde{A}_\nu, \nu \vdash n_0$ are integral as well.

Finally, repeating the above process to the class

$$\widetilde{A}' := \widetilde{A} - \sum_{|\nu|=n_0} \mathfrak{m}_{\nu,E}(\widetilde{A}_\nu) = \sum_{|\nu|<n_0} \mathfrak{m}_{\nu,E}(\widetilde{A}_\nu),$$

we conclude that (i′) and (ii′) hold for all the classes $\widetilde{A}_\nu$. □

This proposition, together with (5.8), enables us to write down an integral basis of $H^*(\widetilde{X}^{[n]})$ whenever an integral basis of the subspace $\mathbb{H}''_{k,\widetilde{X}} \cong \mathbb{H}'_{k,X}$ is known for every $k \le n$. It is also consistent with subsection 5.3, since (5.5) in Remark 5.7 is applicable to the orthogonal direct sum $H^2(\widetilde{X}; \mathbb{Z}) = H^2(X; \mathbb{Z}) \oplus \mathbb{Z} \cdot E$.

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